

Cube Root Expectation of Cube Root Mean of Random Variables

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Abstract – Concepts of cube root expectation of random variable was introduced with formulating its definition and deriving some of its elementary properties in an earlier study. One property of cube root expectation which relates the cube root expectation and the cube root mean of a finite number of random variables has been derived here. Derivation of the property has been presented in this article.

Keywords: Random Variable, Cube Root Expectation, Cube Root Mean, Relationship.

1. INTRODUCTION

In the literature of statistics expectation, the theoretical average of a random variable, is in general defined as the weighted average of its all possible values with their respective probabilities as the corresponding weights which originally was defined on the basis of weighted arithmetic mean [4, 5, 24] terming it as mathematical expectation [3, 15, 16, 18, 22, 23]. Later on, it was termed as arithmetic expectation [6, 8]. Expectation, which plays a significant role in the theory of statistics, is the basic concept of unbiasedness of estimators in the theory of statistical estimation [13, 14, 21]. Later on, several studies had been done on expectation due to its importance in various fields. In some studies, concept of expectation had been introduced and consequently defined using the concepts of geometric mean [4, 5, 25], harmonic mean [4, 5, 26], quadratic mean [1, 2, 5, 11, 12, 17, 27, 28] and cubic mean [1, 5, 12, 20, 28] which were respectively termed as geometric expectation [6, 8], harmonic expectation [6, 8], quadratic expectation [7, 8] and cubic expectation [9]. Recently from the concept of cube root mean [1, 5, 12, 28], one more concept of expectation termed as cube root expectation of random variable was introduced with formulating its definition and deriving some of its elementary properties [10]. However, more properties are yet to be studied. One property of cube root expectation which relates the cube root expectation and the cube root mean of random variables has been derived in this study. Derivation of the property has been presented in this article.

2. CUBE ROOT EXPECTATION

Let us define cube root expectation of random variable as well as of function of random variable.

Cube Root Expectation of Random Variable

Let X be a real valued random variable assuming values whose cube roots exist.

Cube root expectation [10] of X , denoted by $E_{CR}(X)$, is defined as follows:

In the case where X is finite discrete assuming values

$$x_1, x_2, \dots, x_N$$

with probabilities

$$p_1, p_2, \dots, p_N,$$

respectively, $E_{CR}(X)$ is defined by

$$E_{CR}(X) = \left(\sum_{i=1}^N p_i x_i^{\frac{1}{3}} \right)^3 \quad (2.1)$$

In the case where X is denumerable (countable infinite) discrete assuming values

$$x_1, x_2, \dots$$

with probabilities

$$p_1, p_2, \dots$$

respectively, $ECR(X)$ is defined by

$$E_{CR}(X) = \left(\sum_{i=1}^{\infty} p_i x_i^{\frac{1}{3}} \right)^3 \quad (2.2)$$

$$\text{provided } \sum_{i=1}^{\infty} p_i x_i^{\frac{1}{3}} \text{ exists}$$

Again, in the case where X is continuous assuming values in any of the intervals

$$(a, b) \text{ or } [a, b) \text{ or } (a, b] \text{ or } [a, b]$$

with finite or infinite a & b , following probability density function $f(x)$,

$E_{CR}(X)$ is defined by

$$E_{CR}(X) = \left\{ \int_a^b x^{\frac{1}{3}} f(x) dx \right\}^3 \quad (2.3)$$

Cube Root Expectation of Function of Random Variable

Let

$$\varphi(.) = \varphi(X)$$

be a real valued function of X .

Cube root expectation [10] of $\varphi(X)$, denoted by $E_{CR}\{\varphi(X)\}$, is defined as follows:

In the case where X is finite discrete $ECR\{\varphi(X)\}$, is defined by

$$E_{CR}\{\varphi(X)\} = \left[\sum_{i=1}^N p_i \{\varphi(x_i)\}^{\frac{1}{3}} \right]^3 \quad (2.4)$$

In the case where X is denumerable discrete, $E_{CR}\{\varphi(X)\}$ is defined by

$$E_{CR}\{\varphi(X)\} = \left[\sum_{i=1}^{\infty} p_i \{\varphi(x_i)\}^{\frac{1}{3}} \right]^3 \quad (2.5)$$

$$\text{provided } \sum_{i=1}^{\infty} p_i \{\varphi(x_i)\}^{\frac{1}{3}} \text{ exists.}$$

Again, in the case where X is continuous, $E_{CR}\{\varphi(X)\}$ is defined by

$$E_{CR}\{\varphi(X)\} = \left[\int_a^b \{\varphi(x)\}^{\frac{1}{3}} f(x) dx \right]^3 \quad (2.6)$$

Note:

(1) Arithmetic expectation $[6, 8]$ of X , denoted by $E_A(X)$, is defined by

$$E_A(X) = \sum_{i=1}^N p_i x_i \quad (2.7)$$

when X is finite discrete

and by

$$E_A(X) = \sum_{i=1}^{\infty} p_i x_i \quad (2.8)$$

when X is denumerable discrete

$$\text{provided } \sum_{i=1}^{\infty} p_i x_i \text{ exists}$$

and by

$$E_A(X) = \int_a^b x \cdot f(x) dx \quad (2.9)$$

when X is continuous.

Similarly, the arithmetic expectation of $\varphi(X)$, denoted by $E_A\{\varphi(X)\}$, is defined by

$$E_A\{\varphi(X)\} = \sum_{i=1}^N p_i \cdot \varphi(x_i) \quad (2.10)$$

when X is finite discrete

and by

$$E_A\{\varphi(X)\} = \sum_{i=1}^{\infty} p_i \cdot \varphi(x_i) \quad (2.11)$$

when X is denumerable discrete

$$\text{provided } \sum_{i=1}^{\infty} p_i \cdot \varphi(x_i) \text{ exists}$$

and by

$$E_A\{\varphi(X)\} = \int_a^b \varphi(x) f(x) dx \quad (2.12)$$

when X is continuous.

(2) Definitions of arithmetic expectation and cube root expectation implies that

$$E_{CR}(X) = \{E_A(X^{\frac{1}{3}})\}^3 \quad (2.13)$$

i.e. the cube root expectation of a random variable X is the cube of the arithmetic expectation of its cube root.

Similarly, the cube root expectation of a function $\varphi(X)$ of a random variable X is the cube of the arithmetic expectation of the cube root of $\varphi(X)$.

(3) Replacing $X^{\frac{1}{3}}$ by X in (2.13), it is obtained that

$$E_A(X) = \{E_{CR}(X^3)\}^{\frac{1}{3}} \quad (2.14)$$

i.e. the arithmetic expectation of a random variable X is the cube root of the cube root expectation of its cube.

Similarly, the arithmetic expectation of a function $\varphi(X)$ of a random variable X is the cube root of the cube root expectation of the cube of $\varphi(X)$ i.e. of $\{\varphi(X)\}^3$.

3. CUBE ROOT EXPECTATION OF CUBIC MEAN

Let us abbreviate arithmetic mean, cube root mean, arithmetic expectation & cube root expectation by AM, CRM, AE & CRE respectively.

In the Case of Random Variable

Suppose,

$$X_1, X_2, \dots, X_k$$

are k random variables whose arithmetic expectations and cube root expectations exist.

Then from the definitions of cube root expectation and arithmetic expectation of X ,

$$E_{CR}(X_i) = \{E_A(X_i^{\frac{1}{3}})\}^3 \quad (3.1)$$

$$\text{or equivalently } \{E_{CR}(X_i)\}^{\frac{1}{3}} = E_A(X_i^{\frac{1}{3}}) \quad (3.2)$$

($i = 1, 2, \dots, k$).

Replacing X_i by X_i^3 in the above equations, it is obtained that

$$E_{CR}(X_i^3) = \{E_A(X_i)\}^3 \quad (3.3)$$

$$\text{or equivalently } E_A(X_i) = \{E_{CR}(X_i^3)\}^{\frac{1}{3}} \quad (3.4)$$

($i = 1, 2, \dots, k$).

Now, replacing X_i by $\sum_{i=1}^k X_i$ in equation (3.3), it is obtained that

$$E_{CR}\left\{\left(\sum_{i=1}^k X_i\right)^3\right\} = \{E_A(\sum_{i=1}^k X_i)\}^3$$

By additive property of arithmetic expectation [6, 8],

$$E_A(\sum_{i=1}^k X_i) = \sum_{i=1}^k E_A(X_i)$$

Therefore,

$$E_{CR}\left\{\left(\sum_{i=1}^k X_i\right)^3\right\} = \left\{\sum_{i=1}^k E_A(X_i)\right\}^3$$

which implies,

$$\left[E_{CR}\left\{\left(\sum_{i=1}^k X_i\right)^3\right\}\right]^{\frac{1}{3}} = \sum_{i=1}^k E_A(X_i)$$

which further implies,

$$\left[E_{CR}\left\{\left(\sum_{i=1}^k X_i\right)^3\right\}\right]^{\frac{1}{3}} = \sum_{i=1}^k \{E_{CR}(X_i^3)\}^{\frac{1}{3}}$$

Now, replacing X_i by $X_i^{\frac{1}{3}}$ in this equation, it is obtained that

$$\left[E_{CR}\left\{\left(\sum_{i=1}^k X_i^{\frac{1}{3}}\right)^3\right\}\right]^{\frac{1}{3}} = \sum_{i=1}^k \{E_{CR}(X_i)\}^{\frac{1}{3}}$$

which implies,

$$\left[E_{CR}\left\{\left(\sum_{i=1}^k X_i^{\frac{1}{3}}\right)^3\right\}\right]^{\frac{1}{3}} = \sum_{i=1}^k \{E_{CR}(X_i)\}^{\frac{1}{3}}$$

which further implies,

$$E_{CR}\left\{\left(\frac{1}{k} \sum_{i=1}^k X_i^{\frac{1}{3}}\right)^3\right\} = \left[\frac{1}{k} \sum_{i=1}^k \{E_{CR}(X_i)\}^{\frac{1}{3}}\right]^3 \quad (3.5)$$

This means,

$$E_{CR}\{CRM(X_1, X_2, \dots, X_k)\} = CRM\{E_{CR}(X_1), E_{CR}(X_2), \dots, E_{CR}(X_k)\}$$

In the Case of Function of Random Variable

Let

$$\phi_1(X_1) = \phi_1, \phi_2(X_2) = \phi_2, \dots, \phi_k(X_k) = \phi_k$$

be k functions of

$$X_1, X_2, \dots, X_k$$

respectively.

Then from the definitions of cube root expectation and arithmetic expectation of $\varphi(x)$,

$$E_{CR}(\phi_i) = \{E_A(\phi_i^{\frac{1}{3}})\}^3 \quad \text{or equivalently} \quad \{E_{CR}(\phi_i)\}^{\frac{1}{3}} = E_A(\phi_i^{\frac{1}{3}}),$$

($i = 1, 2, \dots, k$).

Proceeding with the same method as in the earlier case, one can obtain that

$$E_{CR}\left\{\left(\frac{1}{k} \sum_{i=1}^k \phi_i^{\frac{1}{3}}\right)^3\right\} = \left[\frac{1}{k} \sum_{i=1}^k \{E_{CR}(\phi_i)\}^{\frac{1}{3}}\right]^3 \quad (3.6)$$

which means,

$$E_{CR}\{CRM(\phi_1, \phi_2, \dots, \phi_k)\} = CRM\{E_{CR}(\phi_1), E_{CR}(\phi_2), \dots, E_{CR}(\phi_k)\}$$

4. CONCLUSION

The property of cube root expectation, derived here, can be summarized in the form of a theorem mentioned as follows:

Theorem:

“Cube root expectation of cube root mean of a finite number of random variables is the cube root mean of the individual cube root expectations of the variables

and

cube root expectation of cube root mean of a finite number of individual functions of respective random variables is the cube root mean of the individual cube root expectations of the respective functions.”

This property/theorem describes a relationship between the cube root expectation and the cube root mean of random variables as well as of functions of random variable. It is to be mentioned that there may exist more properties of cube root expectation which are yet to be identified. This can be a scope of further research on this type of expectation.

One property of cube root expectation which relates the cube root expectation and the cube root mean of random variables has been derived in this study. Derivation of the property has been presented in this article.

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