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Geometric Expectation of Geometric Mean of Random Variables

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Abstract - An interesting property of geometric expectation, whose concept had been introduced on the basis of geometric mean, and which was consequently defined mathematically, has been identified in the current study. The property describes an interesting result on geometric expectation of geometric mean of random variables. Description of the property has been presented in this article.

Keywords: Random Variable, Geometric expectation, Geometric Mean, Property.

1. INTRODUCTION

The theory of mathematical expectation, developed during the period from the middle of the 17th century to the mid-nineteenth with the works mainly by the mathematicians Blaise Pascal, Chevalier de Méré, Pierre de Fermat, Christiaan Huygen, Pierre-Simon Laplace, Pafnuty Chebyshev and W. A. Whitworth [1 , 11 – 14 , 16, 22 , 28], had originally been defined as weighted average of its all possible values with their respective probabilities as corresponding weights [2 , 15 , 17 , 19] which was later termed as arithmetic expectation [4 , 6] since it was defined on the basis of arithmetic mean [24]. As continuation of the development of the theory of expectation, three more concepts of measure of expectation had been introduced and consequently formulated their definitions. These are geometric expectation [4 , 6], harmonic expectation [4 , 6], quadratic expectation [5 , 6] and cubic expectation [9] which are defined based on geometric mean [25], harmonic mean [26], quadratic mean [18 , 27] and cubic mean [3 , 10 , 21] respectively.

Study had also been done of the properties of expectation, and some properties were identified in those attempts [2 , 15 , 17 , 19 , 20 , 23 , 29 , 30]. However, many of them are yet to be studied. An interesting property of geometric expectation has been identified in the current attempt. The property describes an interesting result on geometric expectation of geometric mean of random variables. Description of the property has been presented in this article.

2. GEOMETRIC EXPECTATION

Let X be a positive real valued random variable and φ (X) be a function X.

If X is finite discrete and assumes the values

$$x_1 , x_2 , \dots\dots\dots , x_N$$

with respective probabilities

$$p_1 , p_2 , \dots\dots\dots , p_N ,$$

then the geometric expectation of X, denoted by EG(X), is defined by

$$E_G(X) = (\prod_{i=1}^{N} x_i)^{1/N} \qquad (2.1)$$

[4] while by replacing X by φ (X) in equation (2.1), the geometric expectation of φ (X), denoted by EG {$\varphi(x)$}, can be defined by

$$E_G\{\varphi(x)\} = \{\prod_{i=1}^{n} \varphi(x_i)\}^{1/N} \qquad (2.2)$$

If X is denumerable (i.e. countable infinite) discrete and assumes the values

$$x_1, x_2, \ldots\ldots\ldots$$

with respective probabilities

$$p_1, p_2, \ldots\ldots\ldots$$

then $E_G(X)$ is defined by

$$E_G(X) = \underset{N\to\infty}{Lt} (\prod_{i=1}^{N} x_i)^{1/N} \qquad (2.3)$$

provided $\underset{N\to\infty}{Lt} (\prod_{i=1}^{N} x_i)^{1/N}$ exists

while by replacing X by φ (X) in equation (2.3), E_G {$\varphi(x)$} can be defined by

$$E_G\{\varphi(X)\} = \underset{N\to\infty}{Lt} \{\prod_{i=1}^{n} \varphi(x_i)\}^{1/N} \qquad (2.4)$$

provided $\underset{N\to\infty}{Lt} \{\prod_{i=1}^{n} \varphi(x_i)\}^{1/N}$ exists.

Again, if X is a continuous and assumes values in any of the intervals

(a , b) or [a , b) or (a , b] or [a , b]

where a , b may be finite or infinite,

having probability density function f(x),

then $E_G(X)$ is defined by

$$E_G(X) = \exp\{\int_a^b \log_e x . f(x) dx\} \qquad (2.5)$$

[6] while by replacing X by φ (X) in equation (2.5), E_G {$\varphi(x)$} can be defined by

$$E_G\{\varphi(X)\} = \exp\{\int_a^b \log_e \varphi(x) . f(x) dx\} \qquad (2.6)$$

Note (2.1):

Arithmetic expectation [4 , 6] of X, denoted by EA(X), is defined by

$$E_A(X) = \sum_{i=1}^{N} p_i x_i \qquad (2.7)$$

when X is finite discrete

and by

$$E_A(X) = \sum_{i=1}^{\infty} p_i x_i \qquad (2.8)$$

when X is denumerable discrete

provided $\sum_{i=1}^{\infty} p_i x_i$ exists [4]

and by

$$E_A(X) = \int_a^b x.f(x)dx \qquad (2.9)$$

when X is continuous [6].

Similarly by replacing X by φ(X), the arithmetic expectation of φ(X), denoted by EA{φ(X)}, can be defined by

$$E_A\{\varphi(X)\} = \sum_{i=1}^{N} p_i.\varphi(x_i) \qquad (2.10)$$

when X is finite discrete

and by

$$E_A\{\varphi(X)\} = \sum_{i=1}^{\infty} p_i.\varphi(x_i) \qquad (2.11)$$

when X is denumerable discrete

provided $\sum_{i=1}^{\infty} p_i.\varphi(x_i)$ exists

and by

$$E_A\{\varphi(X)\} = \int_a^b \varphi(x) f(x)dx \qquad (2.12)$$

when X is continuous.

Note (2.1):

Definitions of arithmetic expectation and geometric expectation of X imply that

$$E_G(X) = \exp\{E_A(\log_e X)\} \qquad (2.13)$$

i.e. the geometric expectation of a random variable X is the antilogarithm of the arithmetic expectation of the logarithm of X.

Similarly, the definitions of arithmetic expectation and geometric expectation of φ(X) imply that

$$E_G\{\varphi(X)\} = \exp[E_A\{\log_e \varphi(X)\}] \qquad (2.14)$$

i.e. the geometric expectation of φ(X) is the antilogarithm of the arithmetic expectation of the logarithm of φ(X).

3. GEOMETRIC EXPECTATION OF GEOMETRIC MEAN

Let

$$X_1, X_2, \ldots\ldots\ldots, X_k$$

be k real positive valued random variables.

Putting

$$\varphi(X) = \exp(X)$$

in equation (2,14), it is obtained that

$$E_G\{\exp(X)\} = \exp\{E_A(X)\} \quad \text{or} \quad \log_e[E_G\{\exp(X)\}] = E_A(X)$$

which implies,

$$E_G\{\exp(X_i)\} = \exp\{E_A(X_i)\} \quad \text{or} \quad \log_e[E_G\{\exp(X_i)\}] = E_A(X_i)$$

for i = 1 , 2 , ……… , k .

Accordingly,

$$\log_e[E_G\{\exp(X_1 + X_2 + \ldots\ldots + X_k)\}] = E_A(X_1 + X_2 + \ldots\ldots + X_k)$$

By additive property of arithmetic expectation [3 , 13 , 14 , 16 , 21],

$$E_A(X_1 + X_2 + \ldots\ldots + X_k) = E_A(X_1) + E_A(X_2) + \ldots\ldots + E_A(X_k)$$

Therefore,

$$\log_e[E_G\{\exp(X_1 + X_2 + \ldots\ldots + X_k)\}] = \log_e[E_G\{\exp(X_1)\}] + \log_e[E_G\{\exp(X_2)\}] + \ldots\ldots + \log_e[E_G\{\exp(X_k)\}]$$

This implies,

$$\log_e[E_G\{\prod_{i=1}^{k} \exp(X_i)\}] = \log_e\{\prod_{i=1}^{k} E_G\{\exp(X_i)\}$$

$$\text{i.e. } E_G\{\prod_{i=1}^{k} \exp(X_i)\} = \prod_{i=1}^{k} E_G\{\exp(X_i)\}$$

Replacing $\exp(X_i)$ by Y_i , it is obtained that

$$E_G(\prod_{i=1}^{k} Y_i) = \prod_{i=1}^{k} E_G(Y_i)$$

which implies,

$$E_G(\prod_{i=1}^{k} X_i) = \prod_{i=1}^{k} E_G(X_i)$$

This can be regarded as multiplicative property of geometric expectation [ ].

Now, geometric mean, abbreviated by GM , of

$$X_1, X_2, \ldots\ldots\ldots, X_k$$

is given by

$$G_M \text{ of } X_1, X_2, \ldots\ldots, X_k = (\prod_{i=1}^{k} X_i)^{1/k}$$

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Accordingly,

$$E_G \{(\prod_{i=1}^{k} X_i)^{1/k}\} = \{\prod_{i=1}^{k} E_G(X_i)\}^{1/k}$$

Thus the following theorem is obtained:

Theorem (3.1):

If $X_1, X_2, \ldots\ldots, X_k$ are k real positive valued random variables then the geometric expectation of the geometric mean of $X_1, X_2, \ldots\ldots, X_k$ is the geometric mean of their individual geometric expectations.

Now suppose,

$$\phi 1(X_1) = \phi 1, \phi 2(X_2) = \phi 2, \ldots\ldots, \phi k(X_k) = \phi k$$

are k real valued functions of

$$X_1, X_2, \ldots\ldots, X_k$$

respectively.

Then, applying the multiplicative property of geometric expectation to

$$\phi 1(X_1) = \phi 1, \phi 2(X_1) = \phi 2, \ldots\ldots, \phi k(X_k) = \phi k$$

it is obtained that

$$E_G (\prod_{i=1}^{k} \phi_i) = \prod_{i=1}^{k} E_G(\phi\, i)$$

which implies,

$$E_G \{(\prod_{i=1}^{k} \phi_i)^{1/k}\} = \{\prod_{i=1}^{k} E_G(\phi\, i)\}^{1/k}$$

But

$$G_M \text{ of } \phi 1, \phi 2, \ldots\ldots, \phi k = (\prod_{i=1}^{k} \phi_i)^{1/k}$$

$$\& \; G_M \text{ of } E_G(\phi 1), E_G(\phi 2), \ldots\ldots, E_G(\phi k) = \{\prod_{i=1}^{k} E_G(\phi\, i)\}^{1/k}$$

Therefore,

$$E_G \{GM \text{ of } \phi 1, \phi 2, \ldots\ldots, \phi k\} = G_M \text{ of } E_G(\phi 1), E_G(\phi 2), \ldots\ldots, E_G(\phi k)$$

Thus the following theorem is obtained:

Theorem (3.2):

If $\phi1(X_1)$, $\phi2(X_2)$, ..., ϕk (X_k) are k positive real valued functions of the positive real valued random variables X_1 , X_2 , ..., X_k respectively then the geometric expectation of the geometric mean of $\phi1(X_1)$, $\phi2(X_2)$, ..., ϕk (X_k) is the geometric mean of their individual geometric expectations.

4. NUMERICAL EXAMPLE

Let us take an example of a fair dice-throwing experiment

The 6 possible outcomes in a trial of the experiment which are mutually exclusive, equally likely and exhaustive are

1 , 2 , 3 , 4 , 5 , 6

with respective probabilities 1/6 for each of them.

Let us define two random variables X and Y as follows:

X = 2 more than the number occurred in the throw

and Y = double of the number occurred in the throw.

Then the 6 mutually exclusive and exhaustive possible values assumed by X are

3 , 4 , 5 , 6 , 7 , 8 ,

with respective probabilities 1/6 for each of them.

Similarly, the 6 mutually exclusive and exhaustive possible values assumed by Y are

2 , 4 , 6 , 8 , 10 , 12 ,

with respective probabilities 1/6 for each of them.

In this case,

$$E_G(X) = 5.216930942979164018239801244701 7$$

$$\text{and } E_G(Y) = 5.987590331047817909820321135778 9$$

Now, the Geometric Mean of X & Y, which is $\sqrt{XY}$, assumes 25 mutually exclusive and exhaustive possible values with respective probabilities shown in the brackets with the respective values as follows:

2.4494897427831780981972840747059 (1/36) ,

2.8284271247461900976033774484194 (1/36) ,

3.1622776601683793319988935444327 (1/36) ,

3.4641016151377545870548926830117 (1/18) ,

3.7416573867739413855837487323165 (1/36) ,

4 ,0 (1/18)

4.2426406871192851464050661726291 (1/36) ,

4.4721359549995793928183473374626 (1/36) ,

4.8989794855663561963945681494118 (1/12), 5.2915026221291811810032315072785 (1/36) ,

5.4772255750516611345696978280080 (1/18) ,

5.6568542494923801952067548968388 (1/18) ,

6 ,0 (1/18) ,

6.3245553203367586639977870888654 (1/18) ,

6.4807406984078602309659674360880 (1/36) ,

6.9282032302755091741097853660235 (1/12) ,

7.0710678118654752440084436210485 (1/36) ,

7.4833147735478827711674974646331 (1/36) ,

7.7459666924148337703585307995648 (1/18) ,

8. 0 (1/36) ,

8.3666002653407554797817202578519 (1/36) ,

8.4852813742385702928101323452582 (1/36) ,

8.9442719099991587856366946749251 (1/36) ,

9.1651513899116800131760943874560 (1/36) ,

9.7979589711327123927891362988236 (1/36) .

Accordingly,

$$E_G(\sqrt{XY}) = 5.5889932252532045401058658112216$$

Note that

Geometric Mean of $E_G(X)$ & $E_G(Y)$ = 5.5889932252532045401058658112216 = $E_G(\sqrt{XY})$

5. CONCLUSION

The property obtained here can be summarized as follows:

Geometric expectation of geometric mean of a finite number of positive real valued random variables is the geometric mean of the individual geometric expectations of the variables and geometric expectation of geometric mean of a finite number of functions of random variables is the geometric mean of the individual geometric expectations of functions. Similar properties of arithmetic expectation, harmonic expectation and quadratic expectation were already established in earlier studies [7 , 8]. Here the same has been established in the case of geometric expectation. Further problem, in continuation to this study, is to search for whether other types of expectation carry properties similar to those of arithmetic expectation, geometric expectation and, harmonic expectation and quadratic expectation.

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